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Singularity of Cardinal Interpolation with Shifted Box Splines

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Cardinal interpolation by integer translates of shifted box splines $M_{n,\omega} := M_{nnn}(\cdot + \omega)$ on the three-direction mesh is studied. Let $\Lambda := (-\frac{1}{2}, \frac{1}{2})^2 \cap \{(s, t) : |s-t| < \frac{1}{2}\}$. In a previous work by these authors it was shown that the symbol of $M_{n,\omega}$ does not vanish on the torus T^2 for all ω in the shift region Λ. In this work, it is shown that the symbol of $M_{n,\omega}$ always vanishes somewhere on T^2 if $\omega \in [-\frac{1}{2}, \frac{1}{2}]^2 \backslash \Lambda$. In other words the cardinal interpolation operator corresponding to $M_{n,\omega}$, $\omega \in [-\frac{1}{2}, \frac{1}{2}]^2$, $n = 1, 2, \ldots$, is invertible if and only if $\omega \in \Lambda$. © 1993 Academic Press, Inc.

1. INTRODUCTION AND RESULTS

Let ϕ be a piecewise continuous real-valued function with compact support in $\mathbb{R}^s$, $s \geq 1$, and let

$$S(\phi) := \left\{ \sum_{\mathbf{j} \in \mathbb{Z}^s} a_{\mathbf{j}} \phi(\cdot - \mathbf{j}) : a_{\mathbf{j}} \in \mathbb{R} \right\}.$$

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The problem of *cardinal interpolation* from $S(\phi)$ is to study the existence and uniqueness of a bounded "coefficient" sequence $\{a_j\} \subset \mathbb{R}$ corresponding to any given bounded "data" sequence $\{f_{\mathbf{j}}\} \subset \mathbb{R}$, such that the "spline" function $\sum a_{\mathbf{j}}\phi(\cdot - \mathbf{j})$ from $S(\phi)$ agrees with $\{f_{\mathbf{j}}\}$ on $\mathbb{Z}^s$; that is

$$\sum_{\mathbf{j} \in \mathbb{Z}^s} a_{\mathbf{j}}\phi(\mathbf{i}-\mathbf{j}) = f_{\mathbf{i}}, \qquad \mathbf{i} \in \mathbb{Z}^s. \tag{1.1}$$

This problem is said to be *poised* (or *correct*) if corresponding to any bounded data sequence $\{f_{\mathbf{j}}\}$ there exists a unique bounded coefficient sequence $\{a_{\mathbf{j}}\}$ such that (1.1) is satisfied. The *discrete Fourier transform* $\tilde{\phi}$ of ϕ, defined by

$$\tilde{\phi}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{Z}^s} \phi(\mathbf{j})\, e^{-i\mathbf{j}\cdot\mathbf{x}}, \qquad \mathbf{x} \in \mathbb{R}^s, \tag{1.2}$$

plays a central role in the study of the above problem. (Note that $\tilde{\phi}$ agrees with the restriction on the torus T^s of the *symbol* or *z-transform* $\sum \phi(\mathbf{j})\,\mathbf{z}^{-\mathbf{j}}$ of ϕ.) In fact, *the problem of cardinal interpolation from $S(\phi)$ is poised if and only if* $\tilde{\phi}$ never vanishes (cf. [18, 7, 4]).

There have been a number of results on cardinal interpolation for the case in which ϕ corresponds to the symbol of a univariate B-spline with equally spaces knots. Recently, Jetter, Riemenschneider, and Sivakumar [10] unified many results on univariate cardinal spline interpolation by using some fundamental ideas of Schoenberg [13] concerning the spectrum of shifted B-spline interpolation operators. However, the structure of the symbols associated with multivariate box spline cardinal interpolation operators is much less well developed.

In this paper, we restrict our attention to bivariate box splines on a three-direction mesh. Let $M_{n,n,n}$ denote the centered box spline with directions $(1, 0)$, $(0, 1)$, and $(1, 1)$, each repeated n times (cf. [4, Chap. 2]). We study the shifted box spline

$$M_{n,\omega} := M_{n,n,n}(\cdot + \omega) \tag{1.3}$$

defined by

$$\hat{M}_{n,\omega}(\mathbf{x}) = e^{i\omega\cdot\mathbf{x}} \left(\frac{\sin(x_1/2)}{x_1/2} \frac{\sin(x_2/2)}{x_2/2} \frac{\sin((x_1+x_2)/2)}{(x_1+x_2)/2} \right)^n. \tag{1.4}$$

Here, $\mathbf{x} := (x_1, x_2)$ and ω is in the shift region $I^2 := [-\frac{1}{2}, \frac{1}{2}]^2$. Let

$$\Lambda := (-\tfrac{1}{2}, \tfrac{1}{2})^2 \cap \{(s, t) : |s - t| < \tfrac{1}{2}\}. \tag{1.5}$$

In [8, 15], the following theorem was established.

THEOREM 1.1. *For each* $n = 1, 2, \ldots$, $\tilde{M}_{n,\omega}(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in \mathbb{R}^s$ *and* $\omega \in \Lambda$.

The main result in this paper is that if $\omega \in I^2 \setminus \Lambda$, the symbol associated with the translates of $M_{n,\omega}$ always vanishes at some points in $\mathbb{R}^2$.

THEOREM 1.2. *Let* $\omega \in I^2$ *and* $n \in \mathbb{N}$. *Then* $\tilde{M}_{n,\omega}(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *if and only if* $\omega \in \Lambda$.

This result, together with Theorem 1.1, specifies precisely the region of invertibility for shifted box spline symbols. Thus we partially prove a conjecture given in [12], where our Theorems 1.1 and 1.2 are conjectured to hold even for different multiplicities of the three-direction vectors of the box spline. For the statement of Theorem 1.1 this has been fully achieved recently in [1].

An outline of Theorem 1.2 is given in Section 2. The proof of this theorem naturally breaks into two parts, which are given in Theorem 2.1 as parts (a) and (b). Necessary lemmas and the first part of Theorem 2.1 are proved in Section 3. The second part of the theorem as well as the anomalous case $n = 2$ is derived in Sections 4 and 5.

2. OUTLINE OF THE PROOF

Let $\hat{\phi}$ denote, as usual, the Fourier transform of ϕ. Then by an application of the Poisson Summation Formula (cf. [17, p. 49]), we have

$$\tilde{M}_{n,\omega}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{Z}^s} \hat{M}_{n,\omega}(\mathbf{x} + 2\pi\mathbf{j}) \tag{2.1}$$

with $\hat{M}_{n,\omega}$ given in (1.4). Later we will find it convenient to represent $\mathbf{x} \in \mathbb{R}^2$ as $\mathbf{x} = 2\pi(u, v)$.

We wish to show that for shift parameters $\omega \in I^2 \setminus \Lambda$, the symbol $\tilde{M}_{n,\omega}(\mathbf{x})$ vanishes for some $(u, v) \in I^2$. The set of shift parameters ω is reduced by applying the symmetries

$$\tilde{M}_{n, \pm\omega A}(\mathbf{x}) = \tilde{M}_{n,\omega}(\pm\mathbf{x}A^T), \tag{2.2}$$

where A is taken from $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 1\\ -1 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1 & 0\\ 1 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & -1\\ 1 & 1\end{smallmatrix}\right)$; cf. [2, 15, 18]. In fact, if we let N_1 be the triangle

$$N_1 := \operatorname{conv}((-\tfrac{1}{2}, 0), (-\tfrac{1}{3}, \tfrac{1}{3}), (-\tfrac{1}{4}, \tfrac{1}{4})), \tag{2.3}$$

it can be readily shown that

$$I^2 \setminus \Lambda \subset \bigcup_A (\pm A(N_1) + \mathbb{Z}^2),$$

as shown in Fig. 1. Furthermore we limit ourselved to the triangle Δ (cf. Fig. 2) defined by

$$\Delta := \{(u, v): l_v \leqslant u \leqslant \tfrac{1}{2}, v \in [0, \tfrac{1}{2}]\} \qquad \text{with} \quad l_v = \max\{v, \tfrac{1}{2} - v\}. \tag{2.4}$$

Instead of establishing that $\tilde{M}_{n,\omega}(2\pi u, 2\pi v) = 0$ for some point $(u, v) \in \Delta$, we show that a related series vanishes on Δ which has the same zero structure as $\tilde{M}_{n,\omega}$, namely

$$\begin{aligned} Q_{n,\omega}(u, v) &:= \left(\frac{\pi u}{\sin \pi u} \frac{\pi v}{\sin \pi v} \frac{\pi(1-u-v)}{\sin \pi(u+v)}\right)^n \tilde{M}_{n,\omega}(2\pi u, 2\pi v) \\ &= \sum_{j,k \in \mathbb{Z}} e^{i2\pi(j+u, k+v)\cdot\omega} \left(\frac{u}{u+j}\right)^n \left(\frac{v}{v+k}\right)^n \left(\frac{1-u-v}{u+v+j+k}\right)^n. \end{aligned} \tag{2.5}$$

Our approach is to follow along the lines given in [18]. Therefore we first consider the function

$$g(u, v) := g_n(u, v; \omega) := \operatorname{Im} Q_{n,\omega}(u, v) \tag{2.6}$$

on Δ for fixed $\omega \in N_1$. The following properties of g on any horizontal fiber

$$I_v := [l_v, \tfrac{1}{2}] \times \{v\} \tag{2.7}$$

of Δ are the central part of the proof of our main theorem.

THEOREM 2.1. *The function g defined in* (2.6) *satisfies the following conditions.*

(a) $g(l_v, v) \leqslant 0$ *for all* $0 \leqslant v \leqslant \frac{1}{2}$ *and* $g(\frac{1}{2}, v) > 0$ *for all* $0 < v < \frac{1}{2}$.

(b) *At any interior* $(u, v) \in \Delta$, *either* $\partial g/\partial u > 0$ *or* $\partial^2 g/\partial u^2 > 0$.

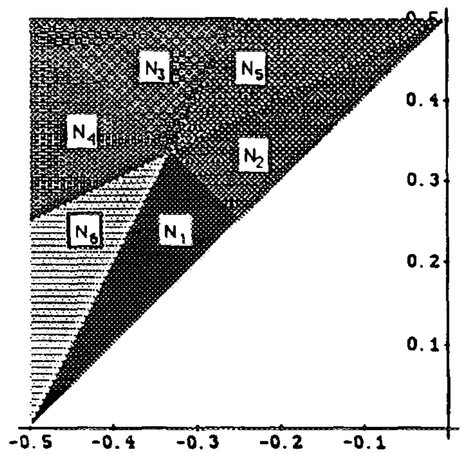

FIG. 1. The shift region N_1.

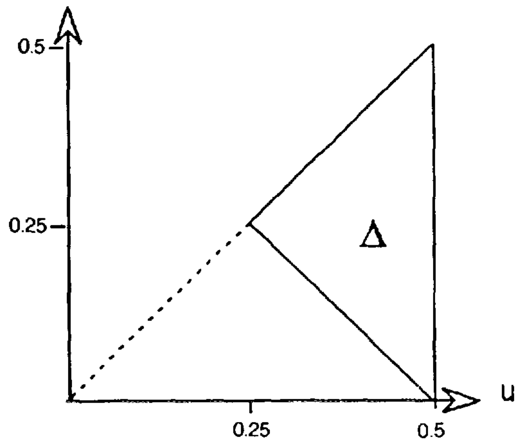

FIG. 2. The angle region Δ.

The above result is proved in Sections 3 and 4 for the case $n \geq 3$ and in Section 5 for $n = 2$. As its conclusion we obtain, as in [18], the following.

THEOREM 2.2. *There is a continuous path of zeros in Δ associated with the family of functions $g(\cdot, v)$. More precisely, we find $s\colon [0, \frac{1}{2}] \to \Delta$ continuous with $s(v) \in I_v$ and $g(s(v)) = 0$ for all $0 \leq v \leq \frac{1}{2}$.*

Proof. By property (b) in Theorem 2.1 the univariate function $g(\cdot, v)$, v fixed, has no local maximum in $(l_v, \frac{1}{2})$. Taking into account the boundary conditions in (a), some elementary considerations yield.

— There is at most one zero of $g(\cdot, v)$ in the open interval $(l_v, \frac{1}{2})$.

— There is at least one zero of $g(\cdot, v)$ in the closed interval $[l_v, \frac{1}{2}]$.

— At any $u \in (l_v, \frac{1}{2})$ we have $g(u, v) = 0 \Rightarrow (\partial g/\partial u)(u, v) > 0$.

Now let

$$s(v) := s_{n,\omega}(v) := \begin{cases} (u, v), & \text{if } u \text{ is an interior zero of } g(\cdot, v) \\ (l_v, v), & \text{if there is no interior zero of } g(\cdot, v). \end{cases}$$

Obviously, $g(s(v)) = 0$ for all $0 \leq v \leq \frac{1}{2}$. Furthermore, if v is given with $s(v) = (u, v)$ and $u > l_v$, then by the Implicit Function Theorem, s is differentiable in a neighbourhood of v. Thus s consists of differentiable arcs whose endpoints have the form (l_{v_1}, v_1), (l_{v_2}, v_2). The above definition of s extends these arcs along the boundary of Δ to a continuous path in Δ. ∎

The proof of Theorem 1.2 will be completed when we show that the function $\operatorname{Re} Q_{n,\omega}$, $\omega \in N_1$, vanishes somewhere along the path s given above. Since we clearly have $s(0)=(\frac{1}{2},0)$ and $s(\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$, this is done by establishing

$$\operatorname{Re} Q_{n,\omega}(\tfrac{1}{2},0)\geqslant 0 \qquad \text{and} \qquad \operatorname{Re} Q_{n,\omega}(\tfrac{1}{2},\tfrac{1}{2})\leqslant 0 \tag{2.8}$$

for all $\omega\in N_1$. Here, the symmetry in (2.2) with $A=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ and the periodicity of $\tilde{M}_{n,\omega}$ prove helpful, since

$$\tilde{M}_{n,\omega}(\pi,\pi)=\tilde{M}_{n,\omega}((\pi,0)\,A^T)=\tilde{M}_{n,\omega A}(\pi,0)=-\tilde{M}_{n,\omega A-(1,0)}(\pi,0).$$

Also, since $A(N_1)-(1,0)=N_6$, it is sufficient to prove the first inequality in (2.8) for all shift parameters in $N_1\cup N_6$.

THEOREM 2.3. *Let $n\geqslant 2$. Then*

$$\operatorname{Re}\tilde{M}_{n,\omega}(\pi,0)\geqslant 0 \qquad \textit{for all} \quad \omega\in N_1\cup N_6.$$

Proof. We prove the inequality for $\omega=(\alpha,\beta)$ in the strip $-\frac{1}{2}\leqslant\alpha\leqslant\frac{1}{2}$. Since $\hat{M}_{n,n,n}((\pi,0)+2\pi(j,k))=0$ if $k\in\mathbb{Z}\backslash\{0\}$, relations (1.4) and (2.1) give

$$\begin{aligned}\operatorname{Re}\tilde{M}_{n,\omega}(\pi,0)&=\operatorname{Re}\sum_{j\in\mathbb{Z}} e^{i\pi(2j+1)\alpha}\hat{M}_{n,n,n}((2j+1)\,\pi,0)\\&=\left(\frac{2}{\pi}\right)^{2n}\sum_{j\in\mathbb{Z}}(2j+1)^{-2n}\cos(2j+1)\,\pi\alpha\\&=2\left(\frac{2}{\pi}\right)^{2n}\sum_{j=0}^{\infty}(2j+1)^{-2n}\cos(2j+1)\,\pi\alpha.\end{aligned}$$

Applying the inequality $|\cos(2j+1)\,\pi\alpha|\leqslant|2j+1|\cos\pi\alpha$ (see Lemma 3.1 below), we see that the last series is bounded from below by

$$\cos\pi\alpha\left(1-\sum_{j=1}^{\infty}(2j+1)^{-3}\right)\geqslant 0.94\cos\pi\alpha\geqslant 0.$$

This completes the proof of the theorem. ∎

3. PROOF OF MAIN RESULT—PART I

We assume $n\geqslant 3$ throughout this section. The case $n=1$ was given in [18] and the case $n=2$ is considered separately in Section 5. In this section we wish to establish part (b) of Theorem 2.1.

We let $\omega = (\alpha, \beta) \in N_1$ denote the shift parameter and we introduce the following notation, which will help to simplify numerous expressions. Thus we set

$$R_{k,l}(u, v) := \frac{u}{u+k}, \qquad S_{k,l}(u, v) := \frac{v}{v+l},$$
$$T_{k,l}(u, v) := \frac{1-u-v}{u+v+k+l}, \qquad \tilde{R}_{k,l}(u, v) := \frac{1-u}{u+k}. \tag{3.1}$$

Furthermore, we define

$$\phi_{k,l}(u, v; \alpha, \beta) := 2\pi(\alpha(u+k) + \beta(v+l)),$$
$$\psi_{k,l}(\alpha, \beta) := 2\pi((k+1)\alpha + l\beta). \tag{3.2}$$

These angles are related via

$$\phi_{k,l} = \phi_{-1,0} + \psi_{k,l}. \tag{3.3}$$

As is done in (3.3), we often drop the arguments of the terms. Setting

$$a_{k,l}(u, v; \alpha, \beta) := \sin \phi_{k,l} (R^n S^n T^n)_{k,l} \tag{3.4}$$

we can represent the function g in (2.6) as

$$g(u, v; \alpha, \beta) = \sum_{k,l \in \mathbb{Z}} a_{k,l}. \tag{3.5}$$

For later reference we give the following partial derivates of $a_{k,l}$ in all points (u, v) interior to Δ. By (3.3) they can be represented in the form

$$\frac{\partial}{\partial u} a_{k,l} = (R^n S^n T^n)_{k,l} \{\sin \phi_{-1,0}[-2\pi\alpha \sin \psi_{k,l} + D_{k,l}^{(n)} \cos \psi_{k,l}]$$
$$+ \cos \phi_{-1,0}[2\pi\alpha \cos \psi_{k,l} + D_{k,l}^{(n)} \sin \psi_{k,l}]\}, \tag{3.6}$$

and

$$\frac{\partial^2}{\partial u^2} a_{k,l} = (R^n S^n T^n)_{k,l} \{\sin \phi_{-1,0}[-4\pi\alpha\, D_{k,l}^{(n)} \sin \psi_{k,l}$$
$$+ (E_{k,l}^{(n)} + F_{k,l}^{(n)} - G_{k,l}^{(n)} - (2\pi\alpha)^2) \cos \psi_{k,l}] + \cos \phi_{-1,0}$$
$$\times [4\pi\alpha\, D_{k,l}^{(n)} \cos \psi_{k,l} + (E_{k,l}^{(n)} + F_{k,l}^{(n)} - G_{k,l}^{(n)} - (2\pi\alpha)^2) \sin \psi_{k,l}]\} \tag{3.7}$$

where

$$D_{k,l}^{(n)} := \frac{nk}{u(u+k)} - \frac{n(k+l+1)}{(1-u-v)(u+v+k+l)},$$

$$E_{k,l}^{(n)} := \frac{n(n-1)\,k^2 - 2nku}{u^2(u+k)^2},$$

$$F_{k,l}^{(n)} := \frac{n(n-1)(k+l+1)^2 + 2n(k+l+1)(1-u-v)}{(1-u-v)^2\,(u+v+k+l)^2},$$

$$G_{k,l}^{(n)} = \frac{2n^2k(k+l+1)}{u(u+k)(1-u-v)(u+v+k+l)}.$$

We wish to stress that all fractions in the above definitions are nonnegative for all possible parameter values. This fact is used in the proofs of Theorem 3.2 and Lemma 3.5.

The following lemma gives some auxiliary inequalities which appear in several places in this and the following section. Parts of it were already mentioned in [15].

LEMMA 3.1. (a) *For any integer* $k \neq 0$ *the function* $u/(u+k)$ *is strictly increasing, and* $(1-u)/(u+k)$ *is strictly decreasing for* $0 \leqslant u \leqslant 1$.

(b) *Given* $n \in \mathbb{N}$, *the functions*

$$f_{\pm}(x, y, z) := (1-x)^{-n}\,(1-y)^{-n}\,(1-z)^{-n} \pm (1+x)^{-n}\,(1+y)^{-n}\,(1+z)^{-n}$$

are strictly increasing in each variable x, y, z *in* $[0, 1)$.

(c) *For any* $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$,

$$|\sin kx| \leqslant |k \sin x|, \qquad |\cos(2k+1)\,x| \leqslant |2k+1|\,|\cos x|.$$

(d) *For any* $x \in (-\pi/2, \pi/2)$,

$$|\sin \varepsilon x| \leqslant |\varepsilon \sin x| \qquad \textit{for} \quad |\varepsilon| \geqslant 1,$$

$$|\tan \varepsilon x| \leqslant |\varepsilon \tan x| \qquad \textit{for} \quad |\varepsilon| \leqslant 1.$$

(e) *For all* $\gamma, \varepsilon \in \mathbb{R}$ *and* $\varepsilon \notin \pi\mathbb{Z}$,

$$|\sin \gamma \pm \cot \varepsilon \cos \gamma| \leqslant \frac{1}{|\sin \varepsilon|}, \qquad |\cos \gamma \pm \cot \varepsilon \sin \gamma| \leqslant \frac{1}{|\sin \varepsilon|}.$$

Proof. Parts (a) and (b) and the first inequality in (c) appear in [15]. The second inequality in (c) follows from the first one and the relation $\cos(2k+1)\,x = \sin((2k+1)\,x + \pi/2) = \pm\sin((2k+1)(x+\pi/2))$. Part (d) is a consequence of the fact that $\sin x$ is concave and $\tan x$ is convex on $[0, \pi/2)$. Part (e) follows from the addition laws of sine and cosine. ∎

Now we prove Theorem 2.1(b) by showing that the function

$$f(u, v; \alpha, \beta) := g(u, v; \alpha, \beta) - a_{0,0}(u, v; \alpha, \beta) \tag{3.8}$$

for a fixed shift parameter (α, β) is monotone increasing and convex on each fiber I_v, and that the single term $a_{0,0}$ as a function of $(u, v) \in \Delta$ satisfies the inequalities $(\partial/\partial u)\, a_{0,0} > 0$ or $(\partial^2/\partial u^2)\, a_{0,0} > 0$.

THEOREM 3.2. *Assume that $(\alpha, \beta) \in N_1$ is fixed. Then for all $n \geq 3$ and any (u, v) interior to Δ, the function f in* (3.8) *has the properties $(\partial/\partial u) f \geq 0$ and $(\partial^2/\partial u^2) f \geq 0$.*

For the proof of Theorem 3.2, the following three lemmas are important. They give bounds for those terms in (3.6), (3.7) which depend only on u and v.

LEMMA 3.3. *The following estimates hold for $(u, v) \in \Delta$, $n, v \in \mathbb{N}$, and $(k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$:*

$$|\tilde{R}^n S^n T^n|_{k,l}$$

$$\leqslant \begin{cases} \left|\dfrac{3}{1+4k}\right|^n \left|\dfrac{1}{1+2l}\right|^n \left|\dfrac{1}{1+2k+2l}\right|^v, & \textit{for} \quad k \neq 0, \quad k+l \neq 0, \\[2ex] \left|\dfrac{3}{1+4l}\right|^n \left|\dfrac{1}{1+2l}\right|^v, & \textit{for} \quad k = 0, \quad l \neq 0, \\[2ex] \left|\dfrac{1}{1+2k}\right|^n \left|\dfrac{1}{1-2k}\right|^n, & \textit{for} \quad k \neq 0, \quad k+l = 0. \end{cases} \tag{3.9}$$

Proof. Note that by Lemma 3.1(a) the terms $|\tilde{R}_{k,l}|$ and $|T_{k,l}|$ in (3.1) are decreasing in u (resp. $u+v$), while $|S_{k,l}|$ is increasing in v. So the first inequality follows from the inequalities $u \geq \frac{1}{4}$, $u + v \geq \frac{1}{2}$, $0 \leq v \leq \frac{1}{2}$. Since $v \leq u$, the second and third inequalities are conclusions of

$$\left|\frac{1-u}{u} \cdot \frac{v}{v+l}\right| \leqslant \left|\frac{1-u}{u} \cdot \frac{u}{u+l}\right| = \left|\frac{1-u}{u+l}\right| \qquad (k=0),$$

$$\left|\frac{1-u}{u+k} \cdot \frac{v}{v-k}\right| \leqslant \left|\frac{1-u}{u+k} \cdot \frac{u}{u+k}\right| = \frac{u(1-u)}{|k^2-u^2|} \leqslant \frac{1}{|4k^2-1|} \qquad (k+l=0). \quad ∎$$

LEMMA 3.4. *For all $n \geq 3$ and $(k, l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ the inequalities*

(i) $|D^{(n)} \tilde{R}^n S^n T^n|_{k,l}$

$$\leq n \max \left\{ \left| \frac{16k}{4k+1} - \frac{4(k+l+1)}{2k+2l+1} \right| |\tilde{R}^n S^n T^n|_{k,l}, \right.$$

$$\left. \frac{4\,|k+l+1|}{(2k+2l+1+\delta_{k+l+1})^2} |\tilde{R}^n S^n T^{n-1}|_{k,l} \right\},$$

(ii) $\frac{u^2(1-u)^2}{n} |E^{(n)} \tilde{R}^n S^n T^n|_{k,l}$

$$\leq \frac{9\,|k|\,((n-1)\,|k| + S_k)}{(4k+1)^2} |\tilde{R}^n S^n T^n|_{k,l},$$

(iii) $\frac{u^2(1-u)^2}{n} |F^{(n)} \tilde{R}^n S^n T^n|_{k,l}$

$$\leq \frac{|k+l+1|\,((n-1)\,|k+l+1| + \delta_{-k-l-1})}{(2k+2l+1+\delta_{k+l+1})^4} |\tilde{R}^n S^n T^{n-2}|_{k,l},$$

(iv) $\frac{u^2(1-u)^2}{n} |G^{(n)} \tilde{R}^n S^n T^n|_{k,l}$

$$\leq \frac{6n\,|k|\,|k+l+1|}{|4k+1|\,(2k+2l+1+\delta_{k+l+1})^2} |\tilde{R}^n S^n T^{n-1}|_{k,l}$$

hold. Here $\delta_v = 1$, if $v < 0$, and $=0$ otherwise. Furthermore the right-hand side in these inequalities has maximal value for $n = 3$, if $(k, l) \neq (0,0)$, $(-1, 0)$, $(0, -1)$.

Proof. Since both fractions in the definition of $D^{(n)}_{k,l}$ are nonnegative, we have

$$|D^{(n)}_{k,l} T_{k,l}| \leq \max \left\{ \left(\max_{u,v} \frac{nk}{u(u+k)} - \min_{u,v} \frac{n(k+l+1)}{(1-u-v)(u+v+k+l)} \right) |T_{k,l}|, \right.$$

$$\left. \times \max_{u,v} \frac{n\,|k+l+1|}{(u+v+k+l)^2} - \min_{u,v} \frac{n\,|k|\,(1-u-v)}{u\,|u+k|\,|u+v+k+l|} \right\}.$$

Using the bounds on $u, v, u+v$ as before leads directly to inequality (i). Analogous arguments are used for (ii) and (iii). That the maximal value in all three inequalities is attained for $n = 3$ is seen by the fact that

$$\frac{n}{n-1} \cdot |\tilde{R}ST|_{k,l} < 1$$

for $n \geqslant 3$ and all $(k, l) \neq (0, 0), (-1, 0), (0, -1)$. Here, we used the bound $|\tilde{R}ST|_{k,l} \leqslant \frac{1}{3}$, which is an immediate consequence of Lemma 3.3. ∎

Using the above inequalities, we also find the following numerical bounds

LEMMA 3.5. *Let $\sum'$ denote summation over $\mathbb{Z}^2 \setminus \{(0, 0), (-1, 0), (0, -1)\}$. It follows that for all $n \geqslant 3$*

$$\sum' |\tilde{R}^n S^n T^n|_{k,l} \leqslant 0.15,$$

$$\sum' |D^{(n)} \tilde{R}^n S^n T^n|_{k,l} \leqslant 1.7,$$

and

$$\sum' |\max\{E^{(n)} + F^{(n)}, G^{(n)} + (2\pi\alpha)^2\} \cdot \tilde{R}^n S^n T^n|_{k,l} \leqslant \frac{0.62n}{u^2(1-u)^2}.$$

Furthermore, the following bounds for the angle $\phi_{-1,0}(u, v; \alpha, \beta)$, where $(u, v) \in \Delta$ and $(\alpha, \beta) \in N_1$, are essential to the proof of Theorem 3.2:

$$\phi_{-1,0} = 2\pi((1-u)|\alpha| + v\beta) \times \begin{cases} \geqslant \pi |\alpha| \geqslant \pi/4, \\ \leqslant 2\pi((1-u)|\alpha| + u\beta) \leqslant \pi/2(3|\alpha| + \beta) \leqslant 3\pi/4. \end{cases} \tag{3.10}$$

Also note that $\psi_{0,-1} = 2\pi(\alpha - \beta) \in [-4\pi/3, -\pi]$ for all $(\alpha, \beta) \in N_1$, and hence

$$\sin \psi_{0,-1} \geqslant 0, \qquad \cos \psi_{0,-1} \leqslant 0. \tag{3.11}$$

Proof of Theorem 3.2. We first wish to establish that $\partial f / \partial u \geqslant 0$. From (3.6) we obtain the special representations

$$\frac{\partial}{\partial u} a_{-1,0} = \left(\frac{u}{1-u}\right)^n \sin \phi_{-1,0} \left[\frac{n}{u(1-u)} + \underbrace{\cot \phi_{-1,0} \cdot 2\pi\alpha}_{=:c_{-1,0}}\right],$$

$$\frac{\partial}{\partial u} a_{0,-1} = \left(\frac{v}{1-v}\right)^n \sin \phi_{-1,0} [\underbrace{-2\pi\alpha \sin 2\pi(\alpha - \beta)}_{=:b_{0,-1}}$$

$$+ \underbrace{\cot \phi_{-1,0} \cdot 2\pi\alpha \cos 2\pi(\alpha - \beta)}_{=:c_{0,-1}}].$$

We show that the term

$$\left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\frac{n}{u(1-u)} \geqslant 12\left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0} \geqslant 0 \qquad (n \geqslant 3) \tag{3.12}$$

which appears in $\partial a_{-1,0}/\partial u$ dominates the entire series $\partial f/\partial u$. First we obtain by (3.10) that

$$1/\sin\phi_{-1,0} \leqslant \sqrt{2}. \tag{3.13}$$

Thus from (3.6) and Lemma 3.1(e) we can conclude that

$$\left(\frac{1-u}{u}\right)^n \left|\frac{\partial a_{k,l}}{\partial u}\right| \leqslant |\tilde{R}^n S^n T^n|_{k,l} \sin\phi_{-1,0} \cdot \sqrt{2}\,(2\pi\,|\alpha| + |D_{k,l}^{(n)}|). \tag{3.14}$$

Now the bounds in Lemma 3.5 give

$$\sum{}' \left|\frac{\partial a_{k,l}}{\partial u}\right| \leqslant \left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0} \cdot \sqrt{2}\,(0.3\pi\,|\alpha| + 1.7). \tag{3.15}$$

In the case $\phi_{-1,0} \in [\pi/4, \pi/2]$, we further observe that $0 \leqslant \cot\phi_{-1,0} \leqslant 1$ and $b_{0,-1} + c_{0,-1} \geqslant 0$ due to (3.11). On the other hand, if $\phi_{-1,0} \in [\pi/2, 3\pi/4]$, we have $-1 \leqslant \cot\phi_{-1,0} \leqslant 0$ and $b_{0,-1} \geqslant 0$ as well as $c_{-1,0} + c_{0,-1} \geqslant 0$. So in both cases we get from (3.12) and (3.15) the lower bound

$$\frac{\partial f}{\partial u} \geqslant \left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}(12 - 2\pi\,|\alpha| - \sqrt{2}\,(0.3\pi\,|\alpha| + 1.7)) \geqslant 0.$$

We now turn to the proof of the inequality $\partial^2 f/\partial u^2 \geqslant 0$. From (3.7), we get the special representations

$$\frac{\partial^2}{\partial u^2} a_{-1,0} = \left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\Bigg(\underbrace{\frac{n(n-1)+2nu}{u^2(1-u)^2} - (2\pi\alpha)^2}_{=:\,d_{-1,0}}$$

$$\underbrace{+\cot\phi_{-1,0} \cdot \frac{4\pi\alpha n}{u(1-u)}}_{=:\,e_{-1,0}}\Bigg),$$

$$\frac{\partial^2}{\partial u^2} a_{0,-1} = \left(\frac{v}{1-v}\right)^n \sin\phi_{-1,0}\big(\underbrace{-(2\pi\alpha)^2 \cos 2\pi(\alpha-\beta)}_{=:\,d_{0,-1}}$$

$$\underbrace{-\cot\phi_{-1,0}(2\pi\alpha)^2 \sin 2\pi(\alpha-\beta)}_{=:\,e_{0,-1}}\big)$$

The dominant term for $\partial^2 f/\partial u^2$ is shown to be

$$\left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\cdot\frac{n}{u^2(1-u)^2}\cdot(n-1+2u)\geqslant 0, \tag{3.16}$$

which again stems from the term $a_{-1,0}$. To prove this we proceed as above and obtain from (3.7), (3.13), and Lemma 3.1.(e)

$$\left(\frac{1-u}{u}\right)^n \left|\frac{\partial^2 a_{k,l}}{\partial u^2}\right| \leqslant |\tilde{R}^n S^n T^n|_{k,l} \sin\phi_{-1,0}$$
$$\times\sqrt{2}\,(4\pi\,|\alpha|\,|D_{k,l}^{(n)}| + \max\{E_{k,l}^{(n)} + F_{k,l}^{(n)}, G_{k,l}^{(n)} + (2\pi\alpha)^2\}).$$

The bounds in Lemma 3.5 give

$$\sum{}' \left|\frac{\partial^2 a_{k,l}}{\partial u^2}\right| \leqslant \left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\cdot\sqrt{2}\left(6.8\pi\,|\alpha| + \frac{0.62n}{u^2(1-u)^2}\right).$$

Under the general assumptions $\frac{1}{4}\leqslant u\leqslant\frac{1}{2}$, $|\alpha|\leqslant\frac{1}{2}$, and $n\geqslant 3$, we have

$$\sum{}' \left|\frac{\partial^2 a_{k,l}}{\partial u^2}\right| \leqslant \left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\frac{\sqrt{2}\,n}{u^2(1-u)^2}\times\left(\frac{1.7\pi\,|\alpha|}{12} + 0.62\right)$$
$$\leqslant 1.2\left(\frac{u}{1-u}\right)^n \sin\phi_{-1,0}\frac{n}{u^2(1-u)^2}. \tag{3.17}$$

We further note that the positive term $d_{0,-1}\geqslant 0$ can be ignored for finding a lower bound of the series. In order to include the possibly negative terms $d_{-1,0}$, $e_{-1,0}$, and $e_{0,-1}$ we again consider two cases. First, if $\phi_{-1,0}\in[\pi/4,\pi/2]$, then (3.10) gives $0\leqslant\cot\phi_{-1,0}\leqslant\cot\pi\,|\alpha|$, and this results in

$$\pi\,|\alpha|\cot\phi_{-1,0}\leqslant\pi\,|\alpha|\cot\pi\,|\alpha|\leqslant\pi/4$$

since the function $z\cot z$ is strictly decreasing in $[\pi/4,\pi/2]$. Hence, in this case we can bound the remaining three terms by

$$\frac{u^2(1-u)^2}{n}(|d_{-1,0}| + |e_{-1,0}| + |e_{0,-1}|)\leqslant\frac{(\pi\alpha)^2}{12} + \frac{\pi}{4}\left(1+\frac{\pi}{12}\right)\leqslant 1.2, \tag{3.18}$$

so that combining (3.17), (3.18), and

$$n-1+2u\geqslant 2.5, \tag{3.19}$$

we have shown that (3.16) dominates the series $\partial^2 f/\partial u^2$.

In the second case, where $\phi_{-1,0}\in(\pi/2, 3\pi/4]$, we can ignore the positive terms $d_{0,-1}$, $e_{-1,0}$, and $e_{0,-1}$ for finding a lower bound of the series. Hence, the desired result is contained in (3.17), (3.18), and (3.19).

This completes the proof of Theorem 3.2. ∎

One can conclude from Theorem 3.2 that the term $a_{0,0}(u, v; \alpha, \beta)$ controls both the monotonicity and the convexity of the entire series

$$g(u, v; \alpha, \beta) = f(u, v; \alpha, \beta) + a_{0,0}(u, v; \alpha, \beta).$$

We proceed next to investigate the monotonicity and convexity properties of the $a_{0,0}$ term. We wish to show that either $(\partial/\partial u)\, a_{0,0}$ or $(\partial^2/\partial u^2)\, a_{0,0}$ is strictly positive for (u, v) interior to Δ and any shift $\alpha, \beta \in N_1$. To this end, recall that by (3.4)

$$a_{0,0} = T^n_{0,0} \sin \phi_{0,0} = \left(\frac{1-u-v}{u+v}\right)^n \sin 2\pi(\alpha u + \beta v).$$

The first second derivatives are respectively

$$\frac{\partial}{\partial u} a_{0,0} = T^n_{0,0}(2\pi\alpha \cos \phi_{0,0} + D^{(n)}_{0,0} \sin \phi_{0,0}),$$

$$\frac{\partial^2}{\partial u^2} a_{0,0} = T^n_{0,0}(4\pi\alpha\, D^{(n)}_{0,0} \cos \phi_{0,0} + (F^{(n)}_{0,0} - (2\pi\alpha)^2) \sin \phi_{0,0}).$$

The terms $\sin \phi_{0,0}$ and $D^{(n)}_{0,0}$ are negative, while $T^n_{0,0}$, $\cos \phi_{0,0}$, and $F^{(n)}_{0,0}$ are positive. Hence, the following string of implications proves the desired properties of $a_{0,0}$:

$$\frac{\partial}{\partial u} a_{0,0} \leqslant 0$$

$$\Rightarrow \frac{n}{(1-u-v)(u+v)} |\sin \phi_{0,0}| = |D^{(n)}_{0,0} \sin \phi_{0,0}| \leqslant 2\pi\, |\alpha| \cos \phi_{0,0}$$

$$\Rightarrow \frac{n-1+2(1-u-v)}{(1-u-v)(u+v)} |\sin \phi_{0,0}| = \frac{F^{(n)}_{0,0}}{|D^{(n)}_{0,0}|} |\sin \phi_{0,0}| \leqslant 2\pi\, |\alpha| \cos \phi_{0,0}$$

$$\Rightarrow (F^{(n)}_{0,0} - (2\pi\alpha)^2)\, |\sin \phi_{0,0}| < 4\pi\, |\alpha|\, |D^{(n)}_{0,0}| \cos \phi_{0,0}$$

$$\Rightarrow \frac{\partial^2}{\partial u^2} a_{0,0} > 0$$

Note that the strict inequality in the third line implications is justified by the strict positivity of the terms on the right-hand side. (The case $\cos \phi_{0,0} = 0$ is excluded by the inequality in the first line.) Thus, the proof of Theorem 2.1(b) is complete.

4. The Boundary Cases

In this section we give a proof of Theorem 2.1(a). We adopt the notations from Section 3. Hence, keeping the shift parameter $(\alpha, \beta) \in N_1$ fixed (c.f (2.3) for the definition of N_1) we wish to show that the function g in (2.6) has the properties

$$g(\tfrac{1}{2}, v) > 0 \quad \text{for all} \quad 0 < v < \tfrac{1}{2}, \tag{4.1}$$

$$g(\tfrac{1}{2} - v, v) \leqslant 0 \quad \text{for all} \quad 0 \leqslant v \leqslant \tfrac{1}{4}, \tag{4.2}$$

$$g(v, v) \leqslant 0 \quad \text{for all} \quad \tfrac{1}{4} \leqslant v \leqslant \tfrac{1}{2}. \tag{4.3}$$

In the present considerations the series representation (3.5) of g is used.

We generally assume $n \geqslant 2$ in the proofs of (4.1) and (4.3), while series representation (3.5) only allows us to prove (4.2) in the case $n \geqslant 3$. The missing part, $n = 2$, is considered in Section 5.

The general trigonometric identity

$$A \sin \gamma + B \sin \varepsilon = (A + B) \sin \frac{\gamma + \varepsilon}{2} \cos \frac{\gamma - \varepsilon}{2} + (A - B) \cos \frac{\gamma + \varepsilon}{2} \sin \frac{\gamma - \varepsilon}{2}, \tag{4.4}$$

which holds for all $A, B, \gamma, \varepsilon \in \mathbb{R}$, is useful for combining terms in the series (3.5).

We now turn to the proof of the first boundary condition.

Proof of (4.1)

Case 1a. We first restrict our attention to $(u, v) = (\frac{1}{2}, v)$ with $0 < v \leqslant \frac{1}{4}$. The monotonicity of the terms R_{kl}, S_{kl}, and T_{kl} in (3.1), which is described in Lemma 3.1(a), yields for all $k, l \in \mathbb{Z}$ with $l \neq 0$

$$\left| a_{kl}\left(\frac{1}{2}, v; \alpha, \beta\right) \right| = |\sin \phi_{kl}| \left(\frac{v}{1-v}\right)^n |2k+1|^{-n} \left|\frac{1-v}{v+l}\right|^n \left|\frac{1-2v}{1+2v+2k+2l}\right|^n$$
$$\leqslant \left(\frac{v}{1-v}\right)^n |2k+1|^{-n} |l|^{-n} |2k+2l+1|^{-n}. \tag{4.5}$$

Summation of these terms for $n \geqslant 2$ gives the bound

$$\sum_{l \neq 0} |a_{kl}| \leqslant 2.7 \left(\frac{v}{1-v}\right)^n \leqslant 2.7 \left(\frac{v}{1-v}\right)^2, \tag{4.6}$$

which is independent of the shift parameters.

Furthermore, by use of the identities

$$\tfrac{1}{2}(\phi_{k,0}+\phi_{-k-1,0})=2\pi v\beta,$$
$$\tfrac{1}{2}(\phi_{k,0}-\phi_{-k-1,0})=(2k+1)\,\pi\alpha,$$

and (4.4), we can combine the terms

$$a_{k,0}+a_{-k-1,0}=b_k+c_k$$

with

$$\begin{aligned} b_k &:= [(R^nS^nT^n)_{-k-1,0}+(R^nS^nT^n)_{k,0}]\sin 2\pi v\beta\cos(2k+1)\,\pi\alpha,\\ c_k &:= -[(R^nS^nT^n)_{-k-1,0}-(R^nS^nT^n)_{k,0}]\cos 2\pi v\beta\sin(2k+1)\,\pi\alpha. \end{aligned} \tag{4.7}$$

Obviously we have

$$\sum_{k\in\mathbb{Z}} a_{k,0}=\sum_{k=0}^{\infty}(b_k+c_k), \tag{4.8}$$

where with the abbreviation $X(v):=(1-2v)/(1+2v)$

$$\begin{aligned} b_0 &= [1+X^n(v)]\sin 2\pi v\beta\cos\pi\alpha\geqslant 0,\\ c_0 &= -[1-X^n(v)]\cos 2\pi v\beta\sin\pi\alpha\geqslant\sqrt{6}/4[1-X^n(v)]\geqslant 0. \end{aligned} \tag{4.9}$$

Now for the leading terms in (4.7) using Lemma 3.1(b) we find that

$$|[(R^nS^nT^n)_{-k-1,0}\pm(R^nS^nT^n)_{k,0}]|\leqslant|2k+1|^{-2n}\,[1\pm X^n(v)].$$

With Lemma 3.1(c), summation over $k\geqslant 0$ gives

$$\sum_{k=0}^{\infty}c_k\geqslant c_0-\sum_{k=1}^{\infty}|c_k|\geqslant c_0-\sum_{k=1}^{\infty}(2k+1)^{-2n+1}\,c_0\geqslant 0.94c_0. \tag{4.10}$$

For the numerical constant 0.94 we assume $n\geqslant 2$, thus

$$\sum_{k=1}^{\infty}(2k+1)^{-2n+1}\leqslant\sum_{k=1}^{\infty}(2k+1)^{-3}\leqslant 0.06. \tag{4.11}$$

By the same string of inequalities, the inequality $\sum_{k=0}^{\infty}b_k\geqslant 0.94b_0$ is established. Now the positivity of the complete series $g(\tfrac{1}{2},v)$ follows from the fact that the expression in (4.10) dominates (4.6). This is seen by

$$c_0\geqslant\frac{\sqrt{6}}{4}(1-X^2(v))=2\sqrt{6}\,v(1+2v)^{-2}\geqslant 2\sqrt{6}\left(\frac{v}{1-v}\right)^2,$$

which respects our general bounds $-\frac{1}{2} \leqslant \alpha \leqslant -\frac{1}{4}$, $0 \leqslant \beta \leqslant \frac{1}{3}$, and $0 < v \leqslant \frac{1}{4}$. Thus we have proved (4.1) for all $0 < v \leqslant \frac{1}{4}$.

Case 1b. The remaining part, $\frac{1}{4} < v < \frac{1}{2}$, is related to Case 1a by the following symmetries. Let $m := -k - l - 1$ and $\varepsilon := \frac{1}{2} - v$ with $0 < \varepsilon < \frac{1}{4}$. Then it is easily checked that

$$(RST)_{k,l}\left(\tfrac{1}{2}, v\right) = (RST)_{k,m}\left(\tfrac{1}{2}, \varepsilon\right),$$

$$\phi_{k,l}\left(\tfrac{1}{2}, v; \alpha, \beta\right) = \phi_{k,m}\left(\tfrac{1}{2}, \varepsilon; \alpha - \beta, -\beta\right).$$

This gives

$$\sum_{k,l \in \mathbb{Z}} a_{k,l}\left(\tfrac{1}{2}, v; \alpha, \beta\right) = \sum_{k,m \in \mathbb{Z}} a_{k,m}\left(\tfrac{1}{2}, \varepsilon; \alpha - \beta, -\beta\right).$$

The sum of all terms in the latter series with $m \neq 0$ is bounded in absolute value by $2.7(\varepsilon/(1-\varepsilon))^n$ as in (4.6). Furthermore, note that for $m = 0$ we have $(a_{k,0} + a_{-k-1,0})(\frac{1}{2}, \varepsilon; \alpha - \beta, -\beta) = \tilde{b}_k + \tilde{c}_k$ with

$$\tilde{b}_0 = -[1 + X^n(\varepsilon)] \sin 2\pi\varepsilon\beta \cos \pi(\alpha - \beta) \geqslant 0,$$

$$\tilde{c}_0 = -[1 - X^n(\varepsilon)] \cos 2\pi\varepsilon\beta \sin \pi(\alpha - \beta) \geqslant \tfrac{3}{4}[1 - X^n(\varepsilon)] \geqslant 0;$$

cf. (4.9). Thus the argument used in Case 1a applies to prove (4.1) for all $\frac{1}{4} < v < \frac{1}{2}$ and $n \geqslant 2$. ∎

Proof of (4.2)

Here we assume $n \geqslant 3$ and split the proof into two different parts.

Case 2a. Let $0 \leqslant v \leqslant \frac{1}{8}$ and $u = \frac{1}{2} - v$. Our procedure is similar to that in Case 1a. For revealing the symmetries we again let $m := -k - l - 1$. Then by simple algebraic operations

$$(RST)_{k,l}\left(\tfrac{1}{2} - v, v\right) = (RST)_{m,l}\left(\tfrac{1}{2}, v\right),$$

$$\phi_{k,l}\left(\tfrac{1}{2} - v, v; \alpha, \beta\right) = \phi_{m,l}\left(\tfrac{1}{2}, v; -\alpha, \beta - \alpha\right).$$

Now the series (3.5) is transformed to

$$\sum_{k,l \in \mathbb{Z}} a_{k,l}\left(\tfrac{1}{2} - v, v; \alpha, \beta\right) = \sum_{m,l \in \mathbb{Z}} a_{m,l}\left(\tfrac{1}{2}, v; -\alpha, \beta - \alpha\right).$$

The sum of all terms with $l \neq 0$ is again bounded in absolute value by

$2.7(v/(1-v))^n$ (cf. (4.6)). For terms with $l=0$ we proceed by combining $(a_{m,0}+a_{-m-1,0})(\frac{1}{2}, v; -\alpha, \beta-\alpha)=b_m^*+c_m^*$ as in (4.8) with

$$\begin{aligned} b_0^* &= [1+X^n(v)]\sin 2\pi v(\beta-\alpha)\cos\pi\alpha \geqslant 0, \\ c_0^* &= [1-X^n(v)]\cos 2\pi v(\beta-\alpha)\sin\pi\alpha \leqslant -\frac{\sqrt{6}}{4}[1-X^n(v)] \leqslant 0. \end{aligned} \tag{4.12}$$

Similar to (4.10), i.e., by use of (4.11), we obtain that

$$\sum_{m\geqslant 0} c_m^* \leqslant 0.94 c_0^*, \qquad \sum_{m\geqslant 0} b_m^* \leqslant 1.06 b_0^*.$$

In order to prove that $g(\frac{1}{2}-v, v)\leqslant 0$ for all $0\leqslant v\leqslant\frac{1}{8}$, it therefore suffices to show that

$$2.7\left(\frac{v}{1-v}\right)^n + 1.06 b_0^* \leqslant 0.94\,|c_0^*|.$$

Since $n\geqslant 3$, we have the relation

$$1-X^n(v)\geqslant 1-X^3(v)=4v(3+4v^2)(1+2v)^{-3},$$

and since $v\leqslant\frac{1}{8}$, it follows that

$$\begin{aligned} &1-X^n(v)\geqslant 5v(2+24v^2)(1+2v)^{-3}=5v(1+X^3(v))\geqslant 5v(1+X^n(v)), \\ &1-X^n(v)\geqslant 263\left(\frac{v}{1-v}\right)^3 \geqslant 263\left(\frac{v}{1-v}\right)^n. \end{aligned} \tag{4.13}$$

Hence, we arrive at the bound

$$2.7\left(\frac{v}{1-v}\right)^n \leqslant \frac{2.7}{263}\cdot\frac{4}{\sqrt{6}}\,|c_0^*| \leqslant 0.018\,|c_0^*|. \tag{4.14}$$

Furthermore, from Lemma 3.1(d) and $0\leqslant\beta\leqslant|\alpha|$, we conclude that

$$0\leqslant\frac{\tan 2\pi v(\beta-\alpha)\cot\pi\,|\alpha|}{5v}\leqslant\frac{4}{5}\cdot\frac{\tan 4\pi v\,|\alpha|}{4v\tan\pi\,|\alpha|}\leqslant\frac{4}{5},$$

and this, together with (4.13), gives

$$1.06 b_0^* \leqslant 0.848\,|c_0^*|. \tag{4.15}$$

Combining (4.14) and (4.15) shows the desired result.

Case 2b. Here $\frac{1}{8} \leqslant v \leqslant \frac{1}{4}$ and $u = \frac{1}{2} - v$. We let $\varepsilon := \frac{1}{4} - v \in [0, \frac{1}{8}]$ and wish to prove that the single term

$$a_{0,0} = a_{0,0}(\tfrac{1}{4} + \varepsilon, \tfrac{1}{4} - \varepsilon; \alpha, \beta) = \sin \phi_{0,0} \leqslant 0 \tag{4.16}$$

dominates the series (3.5). Here, we have

$$\phi_{0,0} = \frac{\pi}{2}(\alpha + \beta) + 2\pi\varepsilon(\alpha - \beta) \in [-3\pi/8, 0]. \tag{4.17}$$

To this end we need a modfied version of the trigonometric inequalities of Lemma 3.1(c), namely; for any $m \in \mathbb{Z}$,

$$\begin{aligned} &\left|\sin\left(\frac{\pi}{2}(2m+1)(\alpha+\beta) + 2\pi\varepsilon(\alpha-\beta)\right)\right| \\ &\qquad \leqslant |2m+1|\, |\sin \phi_{0,0}|, \\ &\left|\cos\left(\frac{\pi}{2}(2m+1)(\alpha+\beta) + 2\pi\varepsilon(\alpha-\beta)\right)\right| \\ &\qquad \leqslant |2m+1| \cos\left(\frac{\pi}{2}(\alpha+\beta) - 2\pi\varepsilon(\alpha-\beta)\right). \end{aligned} \tag{4.18}$$

These inequalities follow immediately from Lemma 3.1(c) by expanding the left-hand side in (4.18) with the addition law. Now, we obtain from (4.18) and Lemma 3.1(a) that for all $k \in \mathbb{Z}\setminus\{0\}$

$$|\sin \phi_{k,k}| = \left|\sin\left(\frac{\pi}{2}(4k+1)(\alpha+\beta) + 2\pi\varepsilon(\alpha-\beta)\right)\right| \leqslant |4k+1|\, |\sin \phi_{0,0}|,$$

$$|R^n S^n T^n|_{k,k} = |4k+1|^{-n} \left|\frac{1-16\varepsilon^2}{(4k+1)^2 - 16\varepsilon^2}\right|^n \leqslant |4k+1|^{-3n}.$$

This gives, by using (4.11) and $n \geqslant 3$, a bound on the first part of the series

$$\begin{aligned} \sum_{k \neq 0} |a_{k,k}| &\leqslant \sum_{k \neq 0} |4k+1|^{-3n+1} |a_{0,0}| = \sum_{k > 0} |2k+1|^{-3n+1} |a_{0,0}| \\ &\leqslant 0.06\, |a_{0,0}|. \end{aligned} \tag{4.19}$$

For $k > l$ we combine the terms

$$a_{k,l} + a_{l,k} = b_{k,l} + c_{k,l}, \qquad k > l,$$

as in indentity (4.4). Hence, with

$$\frac{1}{2}(\phi_{k,l}+\phi_{l,k})=\frac{\pi}{2}(2k+2l+1)(\alpha+\beta)+2\pi\varepsilon(\alpha-\beta),$$

$$\tfrac{1}{2}(\phi_{k,l}-\phi_{l,k})=\pi(k-l)(\alpha-\beta),$$

we have

$$\begin{aligned} b_{k,l} &= [(R^nS^nT^n)_{k,l}+(R^nS^nT^n)_{l,k}] \\ &\quad\times \sin\left(\frac{\pi}{2}(2k+2l+1)(\alpha+\beta)+2\pi\varepsilon(\alpha-\beta)\right)\cos\pi(k-l)(\alpha-\beta), \\ c_{k,l} &= [(R^nS^nT^n)_{k,l}-(R^nS^nT^n)_{l,k}] \\ &\quad\times \cos\left(\frac{\pi}{2}(2k+2l+1)(\alpha+\beta)+2\pi\varepsilon(\alpha-\beta)\right)\sin\pi(k-l)(\alpha-\beta). \end{aligned} \tag{4.20}$$

A bound on the series $\sum_{k>l}|b_{k,l}|$ is obtained from the inequality (4.18) and Lemma 3.1(a) with $u\leqslant\frac{3}{8}$, $v\leqslant\frac{1}{4}$, namely,

$$\begin{aligned} \sum_{k>l}|b_{k,l}| &\leqslant \sum_{k\neq l}|R^nS^nT^n|_{k,l}\,|2k+2l+1|\,|a_{0,0}| \\ &\leqslant \sum_{k\neq l}\left|\frac{3}{8k+3}\right|^n |4l+1|^{-n}\,|2k+2l+1|^{-n+1}\,|a_{0,0}| \\ &\leqslant 0.27\,|a_{0,0}|. \end{aligned} \tag{4.21}$$

The numerical constant 0.27 is found assuming that $n\geqslant 3$. The sum over the terms $c_{k,l}$ in (4.20) will be bounded by a multiple of

$$\begin{aligned} c_{0,-1} &= -\Bigg[\underbrace{\left(\frac{1+4\varepsilon}{3-4\varepsilon}\right)^n}_{=:Y}-\underbrace{\left(\frac{1-4\varepsilon}{3+4\varepsilon}\right)^n}_{=:Z}\Bigg] \\ &\quad\times\cos\left(\frac{\pi}{2}(\alpha+\beta)-2\pi\varepsilon(\alpha-\beta)\right)\sin\pi(\alpha-\beta)\geqslant 0. \end{aligned} \tag{4.22}$$

From Lemma 3.1(b) and the observation

$$\begin{aligned} &|(1+x)^{-n}(1+y)^{-n}\pm(1-x)^{-n}(1-y)^{-n}| \\ &\quad\leqslant(1-|x|)^{-n}(1-|y|)^{-n}\pm(1+|x|)^{-n}(1+|y|)^{-n}, \end{aligned}$$

which holds for all $-1 \leqslant x,\ y \leqslant +1$, we conclude that for $k \neq l$

$$\begin{aligned} &|(4k+1+4\varepsilon)^{-n}(4l+1-4\varepsilon)^{-n}-(4k+1-4\varepsilon)^{-n}(4l+1+4\varepsilon)^{-n}| \\ &\quad \leqslant 3^n\,|4k+1|^{-n}\,|4l+1|^{-n} \\ &\qquad \times ((3-4\varepsilon)^{-n}(1-4\varepsilon)^{-n}-(3+4\varepsilon)^{-n}(1+4\varepsilon)^{-n}). \end{aligned}$$

This and (4.18) give, for all $k > l$,

$$|c_{k,l}| \leqslant (k-l)\,|2k+2l+1|^{-n+1}\,3^n\,|4k+1|^{-n}\,|4l+1|^{-n}\,c_{0,-1},$$

and summation over all $k > l$ leads to

$$\sum_{k>l} |c_{k,l}| \leqslant 1.1 c_{0,-1}. \qquad (4.23)$$

Finally, we need to find a bound on $c_{0,-1}$ in terms of $|a_{0,0}|$. First observe that $c_{0,-1}$ decreases in n. This fact is based on the relations $0 < Z \leqslant Y$ and $Y+Z<1$ in (4.22), because then

$$Y^n - Z^n > (Y+Z)(Y^n-Z^n) = Y^{n+1} - Z^{n+1} + \underbrace{YZ(Y^{n-1}-Z^{n-1})}_{\geqslant 0}.$$

By expanding terms we thus obtain for all $n \geqslant 3$

$$Y^n - Z^n \leqslant Y^3 - Z^3 \leqslant 1.8\varepsilon.$$

Furthermore, for all shift parameters $(\alpha, \beta) \in N_1$ the term $|\sin 2\pi\varepsilon(\alpha-\beta)/\sin \pi(\alpha-\beta)|$ is minimal for $\alpha - \beta = -\frac{1}{2}$; thus, by (4.17) it follows that

$$\left|\frac{\sin \phi_{0,0}}{\sin \pi(\alpha-\beta)}\right| \geqslant \left|\frac{\sin 2\pi\varepsilon(\alpha-\beta)}{\sin \pi(\alpha-\beta)}\right| \geqslant \sin \pi\varepsilon.$$

Since the sine function is concave on $[0, \pi/8]$ we also have

$$\sin \pi\varepsilon \geqslant 8\varepsilon \sin\frac{\pi}{8} \geqslant 3\varepsilon.$$

This gives

$$\frac{c_{0,-1}}{|a_{0,0}|} \leqslant (Y^3 - Z^3)\left|\frac{\sin \pi(\alpha-\beta)}{\sin \phi_{0,0}}\right| \leqslant \frac{1.8\varepsilon}{3\varepsilon} = 0.6.$$

Collecting all terms in (4.19), (4.21), and (4.23), we obtain

$$\left|\sum_{(k,l)\neq(0,0)} a_{k,l}\right| \leqslant (0.06 + 0.27 + 0.66)\,|a_{0,0}| = 0.99\,|a_{0,0}|.$$

Hence the total series g has the same sign as the single term $a_{0,0}$ which was proved to be negative in (4.16). This completes the proof of (4.2). ▌

Proof of (4.3). Here $u=v\in[\frac{1}{4},\frac{1}{2}]$. Since $(RST)_{k,l}(v,v)=(RST)_{l,k}(v,v)$, we obtain as a special case of identity (4.4)

$$a_{k,l}+a_{l,k}=b_{k,l}, \qquad k>l,$$

with

$$\begin{aligned} b_{k,l}=2(R^nS^nT^n)_{k,l}\sin\pi(k+l+2v)(\alpha+\beta) \\ \times\cos\pi(k-l)(\alpha-\beta), \qquad k>l, \end{aligned} \tag{4.24}$$

In particular, we have

$$b_{0,-1}=-2\left(\frac{v}{1-v}\right)^n\sin\pi(1-2v)(\alpha+\beta)\cos\pi(\alpha-\beta)\leqslant 0,$$

and this term dominates $\sum_{k-l\,\mathrm{odd}}a_{k,l}$. To see this, note that by Lemma 3.1(a), (c), and (d), we have for odd values of $k-l$,

$$\begin{aligned} \frac{|b_{k,l}|}{|b_{0,-1}|} &\leqslant |k-l|\left|\frac{k+l+2v}{1-2v}\right|\left(\frac{1-v}{v}\right)^n|R^nS^nT^n|_{k,l} \\ &\leqslant |k-l|\;|2k+2l+1|^{-n+1}\left(\frac{1-v}{v}\right)^n|R^nS^n|_{k,l}. \end{aligned}$$

Using the monotonicity in Lemma 3.1(a) and the inequality $\frac{1}{4}\leqslant v\leqslant\frac{1}{2}$ the bound

$$\begin{aligned} &\left(\frac{1-v}{v}\right)^n|R^nS^n|_{k,l}(v,v) \\ &=\left|\frac{v(1-v)}{(k+v)(l+v)}\right|^n\leqslant\begin{cases}|2k+1|^{-n}\left|\dfrac{3}{4l+1}\right|^n & \text{if}\quad |l|>|k|,\\ |2l+1|^{-n}\left|\dfrac{3}{4k+1}\right|^n & \text{if}\quad |l|<|k|\end{cases} \end{aligned}$$

is established. With $n\geqslant 2$ we therefore find

$$\sum_{k-l\,\mathrm{odd}}a_{k,l}=\sum_{\substack{k-l\,\mathrm{odd}\\ k>l}}b_{k,l}\leqslant b_{0,-1}\left(1-\sum_{\substack{k-l\,\mathrm{odd}\\ k>l}}\frac{|b_{k,l}|}{|b_{0,-1}|}\right)\leqslant 0.3b_{0,-1}\leqslant 0. \tag{4.25}$$

In order to prove that the remaining series

$$\sum_{k-l\,\mathrm{even}}a_{k,l}=\sum_{k\in\mathbb{Z}}a_{k,k}+\sum_{\substack{k-l\,\mathrm{even}\\ k>l}}b_{k,l} \tag{4.26}$$

is negative, we proceed by combining terms in the series. According to identity (4.4), we have

$$a_{k,k} + a_{-k-1,-k-1} = c_{k,k} + d_{k,k},$$
$$b_{k,l} + b_{-l-1,-k-1} = 2(c_{k,l} + d_{k,l}), \qquad k > l,$$

with

$$c_{k,l} := -\sigma_{k,l}[(R^n S^n T^n)_{k,l} + (R^n S^n T^n)_{-l-1,-k-1}]$$
$$\times \sin \pi(1-2v)(\alpha+\beta) \cos \pi(k+l+1)(\alpha+\beta),$$
$$d_{k,l} := \sigma_{k,l}[(R^n S^n T^n)_{k,l} - (R^n S^n T^n)_{-l-1,-k-1}]$$
$$\times \cos \pi(1-2v)(\alpha+\beta) \sin \pi(k+l+1)(\alpha+\beta)$$

and $\sigma_{k,l} = \cos \pi(k-l)(\alpha-\beta)$. The series (4.26) is thus transformed into

$$\sum_{k-l \text{ even}} a_{k,l} = \sum_{k \geqslant 0} (c_{k,k} + d_{k,k}) + \sum_{\substack{k-l \text{ even} \\ k>l,\, k>-l-1}} 2(c_{k,l} + d_{k,l}). \tag{4.27}$$

With $W_\pm := [1 \pm (-1)^n (v/(1-v))^{3n}]$ special values for $k = l = 0$ are

$$c_{0,0} = -W_+ \left(\frac{1-2v}{2v}\right)^n \sin \pi(1-2v)(\alpha+\beta) \cos \pi(\alpha+\beta) \geqslant 0,$$

$$d_{0,0} = W_- \left(\frac{1-2v}{2v}\right)^n \cos \pi(1-2v)(\alpha+\beta) \sin \pi(\alpha+\beta) \leqslant 0.$$

In order to show that (4.27) is dominated by $d_{0,0}$, and thus negative, we need the relation

$$\left|\frac{c_{0,0}}{d_{0,0}}\right| \leqslant \left|\frac{\tan \pi(1-2v)(\alpha+\beta)}{\tan \pi(\alpha+\beta)}\right| \frac{(1-v)^{3n} + v^{3n}}{(1-v)^{3n} - v^{3n}}$$
$$\leqslant (1-2v) \frac{(1-v)^6 + v^6}{(1-v)^6 - v^6} \leqslant \frac{3^6+1}{2(3^6-1)} < 0.51. \tag{4.28}$$

In this string of inequalities we use Lemma 3.1(d) and the fact that the first term in the second line is maximal for $v = \frac{1}{4}$.

Now, letting $\varepsilon := \frac{1}{2} - v \in [0, \frac{1}{4}]$ we obtain from Lemma 3.1(b)

$$|(R^n S^n T^n)_{k,l} \pm (R^n S^n T^n)_{-l-1,-k-1}|$$
$$\leqslant W_\pm \left(\frac{1-2v}{2v}\right)^n |2k+1|^{-n} |2l+1|^{-n} |k+l+1|^{-n}.$$

Thus in conjuction with Lemma 3.1(c) (note that $k+l+1$ is odd) we find

$$|c_{k,l}| \leqslant |2k+1|^{-n}\,|2l+1|^{-n}\,|k+l+1|^{-n+1}\,c_{0,0},$$
$$|d_{k,l}| \leqslant |2k+1|^{-n}\,|2l+1|^{-n}\,|k+l+1|^{-n+1}\,|d_{0,0}|.$$

Summation as in (4.27) now gives for all $n \geqslant 2$

$$\sum_{k>0} |c_{k,k}| + \sum_{\substack{k-l\ \text{even} \\ k>l,\,k>-l-1}} 2\,|c_{k,l}| \leqslant 0.3c_{0,0},$$

$$\sum_{k>0} |d_{k,k}| + \sum_{\substack{k-l\ \text{even} \\ k>l,\,k>-l-1}} 2\,|d_{k,l}| \leqslant 0.3\,|d_{0,0}|.$$

Finally, from (4.27), (4.28), and the last inequalities, we obtain

$$\sum_{k-l\ \text{even}} a_{k,l} \leqslant 0.7\,d_{0,0} + 1.3c_{0,0} \leqslant d_{0,0}(0.7 - 1.3 \cdot 0.51) \leqslant 0.$$

This completes the proof of (4.3). ∎

5. The Case $n=2$

In Section 4 we completed the proof of Theorem 2.1 for the case $n \geqslant 3$. The case $M_{1,1,1}$ was already handled by Stöckler in [18], so there remains only the case $n=2$. For $n=2$, parts (4.1) and (4.3) of the boundary assertions for the function g in (2.6) were already proved in Section 4. For a proof of relation (4.2) and the property of g given in Theorem 2.1(b) we necessarily forsake the series approach since many of our estimates fail for $n=2$ and instead rely on the "low order" approach of Stöckler [18].

We perform the same steps as those used for the noncorrectness proof of cardinal interpolation by verifying Theorem 2.1(a) and (b) when $\phi=\phi_{\alpha,\beta}=M_{2,2,2}(\cdot+\omega)$. Instead of using shift parameters $\omega=(\alpha,\beta)\in N_1$ (cf. (2.3)), we work here with ω in the larger triangle

$$T_1 := \operatorname{conv}\left(\left(-\frac{1}{2},0\right),\left(-\frac{1}{2},\frac{1}{2}\right),\left(-\frac{1}{4},\frac{1}{4}\right)\right), \tag{5.1}$$

which is one half of the triangle T; cf. Fig. 3. Since noncorrectness for all shift parameters ω on the boundary lines of T was observed in [15, 18] we restrict all subsequent work to the intersection of T_1 with the interior of T, thus to

$$K := \{(\alpha,\beta)\colon -\tfrac{1}{2}<\alpha<-\tfrac{1}{4},\ \tfrac{1}{2}-|\alpha|<\beta\leqslant|\alpha|\}. \tag{5.2}$$

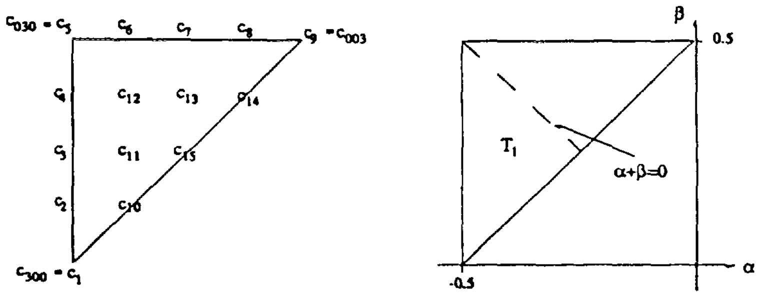

FIG. 3. The triangle T and the Bézier coefficients.

Instead of using the Poisson summation formula, we work with the exact representation of the symbol

$$\tilde{\phi}_{\alpha,\beta}(u, v) = \sum_{k,l \in \mathbb{Z}} M_{2,2,2}(\alpha - k, \beta - l)\, e^{2\pi i(ku + lv)} \tag{5.3}$$

using the Bézier form of the polynomial pieces of $M_{2,2,2}$. Let us first settle on the notations. The Bézier form of a polynomial p of degree 4 with respect to the traingle T is given by

$$p(x, y) = \sum_{0 \leqslant r+s+t \leqslant 4} c_{r,s,t}\, p_{r,s,t}(x, y), \qquad (x, y) \in \mathbb{R}^2,$$

where

$$p_{r,s,t}(x, y) = \frac{4!}{r!\, s!\, t!} (1 - 2y)^r (2y - 2x - 1)^s (1 + 2x)^t$$

are the Bézier basis polynomials related to T. By arranging the coefficients $c_{r,s,t}$ according to the numeration given in Fig. 3 we then write

$$p \sim (c_1, c_2, \ldots, c_{15}).$$

Some basic properties of the basis polynomials show that

$p > 0$ in the interior of T,

$$\text{if } c_k \geqslant 0 \text{ of all } 1 \leqslant k \leqslant 15 \text{ and } c_k > 0 \text{ for at least one } k, \tag{5.4}$$

$p \geqslant 0$ in K with equality if and only if $\alpha + \beta = 0$,

$$\text{if } c_{5+k} = -c_{5-k} \leqslant 0,\ c_{12+l} = -c_{12-l} \leqslant 0\ (0 \leqslant k \leqslant 4,\ 0 \leqslant l \leqslant 2),$$
$$c_{15} = 0, \text{ and } c_k > 0 \text{ for at least one } k \in \{1, 2, 3, 4, 10, 11\}. \tag{5.5}$$

Table I, which gives the Bézier representation for the polynomial pieces of $384 M_{2,2,2}(\alpha-k, \beta-l)$ used in (5.3), where $(\alpha, \beta) \in T$ is assumed, is the basis for our proof. It can be computed by using the recursive algorithm in [4] and the subdivision scheme [9].

TABLE I

The Bézier Coefficients of $M_{2,2,2}(\alpha-k, \beta-l)$ with $(\alpha, \beta) \in T$

k,	l	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
0,	−1	6	1	0	0	0	0	0	1	2	5	1	0	1	3	4
−1,	−1	52	34	20	11	6	7	8	7	6	34	22	13	13	11	20
−2,	−1	6	5	4	3	2	1	0	0	0	1	1	1	0	0	0
1,	0	2	1	0	0	0	0	0	1	6	3	1	0	1	5	4
0,	0	126	113	92	70	52	70	92	113	126	139	119	94	119	139	144
−1,	0	126	139	144	139	126	113	92	70	52	113	119	119	94	70	92
−2,	0	2	3	4	5	6	1	0	0	0	1	1	1	0	0	0
1,	1	6	7	8	7	6	11	20	34	52	11	13	13	22	34	20
0,	1	52	70	92	113	126	139	144	139	126	70	94	119	119	113	92
−1,	1	6	11	20	34	52	34	20	11	6	7	13	22	13	7	8
1,	2	0	0	0	1	2	3	4	5	6	0	0	1	1	1	0
0,	2	0	0	0	1	6	5	4	3	2	0	0	1	1	1	0

We start by proving the boundary assertion (4.2) for the function

$$h(u, v; \alpha, \beta) := 384 \operatorname{Im} \tilde{\phi}_{\alpha,\beta}(u, v), \qquad (\alpha, \beta) \in K, \quad (u, v) \in \Delta, \tag{5.6}$$

which has the same sign structure as that of g in (2.6). This completes the proof of part (a) of Theorem 2.1. In order to prove (4.2) we let $u = \frac{1}{2} - v$ and $0 \leqslant v$ and $0 \leqslant v \leqslant \frac{1}{4}$. Then we obtain from Table I

$$\begin{aligned} h(\tfrac{1}{2}-v, v) &= 384 \sum_{k,l \in \mathbb{Z}} (-1)^k M_{2,2,2}(\alpha-k, \beta-l) \sin 2\pi(l-k) v \\ &= -q_1(\alpha, \beta) \sin 2\pi v - q_2(\alpha, \beta) \sin 4\pi v, \end{aligned}$$

where

$$q_1 \sim (72, 64, 48, 24, \; 0, -24, -48, -64, -72, 44, 24, \; 0, -24, -44, 0),$$

$$q_2 \sim (\;4, \;8, 16, 28, 40, \; 28, \; 16, \; 8, \; 4, \;6, 12, 20, \; 12, \; 6, 8).$$

By (5.4), (5.5) we know that q_1 is positive on the subtriangle T_1 and vanishes for $\alpha+\beta=0$, while $q_2 > 0$ on T. This shows $h(\frac{1}{2}-v, v) \leqslant 0$ with equality if and only if $v=0$, or $v=\frac{1}{4}$ and $\alpha+\beta=0$. Thus (4.2) is also confirmed in case $n=2$.

The last step in our proof is to confirm Theorem 2.1(b) for $n=2$ and for the function h in (5.6). Using Table 1 we find that

$$\begin{aligned}(2\pi)^{-1}\frac{\partial h(u,v)}{\partial u} &= 384 \sum_{k,l\in\mathbb{Z}} kM_{2,2,2}(\alpha-k,\beta-l)\cos 2\pi(ku+lv)\\ &= -s_1\cos 2\pi u - s_2\cos 4\pi u - s_3\cos 2\pi(u+v)\\ &\quad - s_4\cos 2\pi(u-v) - s_5\cos 2\pi(2u+v)\\ &\quad + s_6\cos 2\pi(u+2v),\end{aligned} \tag{5.7}$$

where

$s_1 \sim (124, 138, 144, 139, 126, 113, 92, 69, 46, 110, 118, 119, 93, 65, 88)$,

$s_2 \sim (4, 6, 8, 10, 12, 2, 0, 0, 0, 2, 2, 2, 0, 0, 0)$,

$s_3 \sim (46, 27, 12, 4, 0, -4, -12, -27, -46, 23, 9, 0, -9, -23, 0)$,

$s_4 \sim (6, 11, 20, 34, 52, 34, 20, 11, 6, 7, 13, 22, 13, 7, 8)$,

$s_5 \sim (12, 10, 8, 6, 4, 2, 0, 0, 0, 2, 2, 2, 0, 0, 0)$,

$s_6 \sim (0, 0, 0, 1, 2, 3, 4, 5, 6, 0, 0, 1, 1, 1, 0)$.

Expanding terms in (5.7) using the trigonometric addition laws we obtain

$$(2\pi)^{-1}\frac{\partial h(u,v)}{\partial u} = -\cos 2\pi u\cdot A(\alpha,\beta) + \sin 2\pi u\cdot B(\alpha,\beta)$$

with

$$A(\alpha,\beta) = s_1 + s_2\cos 2\pi u + (s_3+s_4)\cos 2\pi v + s_5\cos 2\pi(u+v) - s_6\cos 4\pi v,$$

$$B(\alpha,\beta) = s_2\sin 2\pi u + (s_3-s_4)\sin 2\pi v + s_5\sin 2\pi(u+v) - s_6\sin 4\pi v.$$

For the second partial derivative we similarly compute

$$\begin{aligned}(2\pi)^{-2}\frac{\partial^2 h(u,v)}{\partial u^2} &= s_1\sin 2\pi u + 2s_2\sin 4\pi u + s_3\sin 2\pi(u+v)\\ &\quad + s_4\sin 2\pi(u-v)\\ &\quad + 2s_5\sin 2\pi(2u+v) - s_6\sin 2\pi(u+2v)\\ &= (A + 2s_2\cos 2\pi u + s_5\cos 2\pi(u+v))\sin 2\pi u\\ &\quad + (B + s_5\sin 2\pi(u+v))\cos 2\pi u.\end{aligned}$$

Again, from (5.4) we conclude that $A > 0$ and $A + 2s_2 \cos 2\pi u + s_5 \cos 2\pi(u+v) > 0$ in K for all, $u, v \in \Delta$. Let us assume that we can find α, β, u, and v with

$$\frac{\partial h(u, v)}{\partial u} \leqslant 0 \qquad \text{and} \qquad \frac{\partial^2 h(u, v)}{\partial u^2} \leqslant 0.$$

Then (5.7) gives

$$B(\alpha, \beta) \leqslant 0, \tag{5.9}$$

while (5.8) yields

$$B(\alpha, \beta) + s_5(\alpha, \beta) \sin 2\pi(u+v) \geqslant 0. \tag{5.10}$$

Since the term $s_5 \sin 2\pi(u+v)$ is nonpositive for all parameters, both (5.9) and (5.10) can hold only if

$$B(\alpha, \beta) = 0 \qquad \text{and} \qquad \sin 2\pi(u+v) = 0.$$

But then we have

$$(2\pi)^{-1} \frac{\partial h(u, v)}{\partial u} = -A \cos 2\pi u \leqslant 0 \qquad \text{iff} \quad u = \frac{1}{4},$$

$$(2\pi)^{-2} \frac{\partial^2 h(u, v)}{\partial u^2} = (A + 2s_2 \cos 2\pi u + s_5 \cos 2\pi(u+v)) \sin 2\pi u$$

$$\leqslant 0 \qquad \text{iff} \quad u = \frac{1}{2},$$

hence a contradiction to our assumption. ∎

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